Chapter 2:
Matrix Algebra

Sec. 2.5: Elementary Matrices

Elementary Matrices

<u>What's the point?</u>

- Row operations are important to solving systems of linear equations
- We just learned how to multiply matrices
- We can perform row operations on a matrix by instead multiplying it on the left by an elementary matrix

Elementary Matrices

Def: An elementary matrix is a matrix obtained by performing EXACTLY ONE elementary row operation on the identity matrix.

Type I	Type II	Type III
Interchange 2 rows	Multiply a row by a nonzero number	Multiply a row by a number, then add to another row

Ex's:

Elementary Matrices

Lemma 2.5.1: [10]

If an elementary row operation is performed on an $m \times n$ matrix A, the result is EA where E is the elementary matrix obtained by performing the same operation on the $m \times m$ identity matrix.

Proof:

Elementary Matrices

Lemma 2.5.2

Every elementary matrix E is invertible, and E^{-1} is also a elementary matrix (of the same type). Moreover, E^{-1} corresponds to the inverse of the row operation that produces E.

Type	**Operation**	**Inverse Operation**
I	Interchange rows p and q	Interchange rows p and q
II	Multiply row p by $k \neq 0$	Multiply row p by $1/k$, $k \neq 0$
III	Add k times row p to row $q \neq p$	Subtract k times row p from row q, $q \neq p$

Elementary Matrices

Ex 1:

Find the inverse of each of the elementary matrices

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{bmatrix}, \quad \text{and} \quad E_3 = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Elementary Matrices

If matrix A can be carried to matrix B ($A \rightarrow B$) by a sequence of elementary row operations, then $B = E_k E_{k-1} \cdots E_2 E_1 A$ where each E_i is an elementary matrix.

Elementary Matrices

Theorem 2.5.1

Suppose A is $m \times n$ and $A \to B$ by elementary row operations.

1. *$B = UA$ where U is an $m \times m$ invertible matrix.*
2. *U can be computed by $\left[\begin{array}{cc} A & I_m \end{array} \right] \to \left[\begin{array}{cc} B & U \end{array} \right]$ using the operations carrying $A \to B$.*
3. *$U = E_k E_{k-1} \cdots E_2 E_1$ where $E_1,\ E_2,\ \dots,\ E_k$ are the elementary matrices corresponding (in order) to the elementary row operations carrying A to B.*

Elementary Matrices

Ex 2:

If $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix}$, express the reduced row-echelon form R of A as $R = UA$ where U is invertible.

Elementary Matrices

Theorem 2.5.2

A square matrix is invertible if and only if it is a product of elementary matrices.

Proof:

Elementary Matrices

<u>Ex 3</u>:

Express $A = \begin{bmatrix} -2 & 3 \\ 1 & 0 \end{bmatrix}$ as a product of elementary matrices.